

# Geometric optics for a coupling model of electromagnetic and gravitational fields

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## Abstract

The coupling between the electromagnetic and gravitational fields results in “faster than light” photons and invalids the Lorentz invariance and some laws of physics. A typical example is that the first and third laws of geometric optics are invalid in the usual spacetime. By introducing an effective spacetime, we find that the wave vector can be casted into null and then it obeys the geodesic equation, the polarization vector is perpendicular to the rays, and the number of photons is conserved. That is to say, the laws of geometric optics are still valid for the modified theory in the effective spacetime. We also show that the focusing theorem of light rays for the modified theory in the effective spacetime takes the same form as usual.

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## I. INTRODUCTION

It is well known that the propagation of light and radio waves are subject to the laws of geometric optics. The fundamental laws of geometric optics are: (1) light rays are null geodesics; (2) the polarization vector is perpendicular to the rays and parallel-propagated along the rays; and (3) the amplitude is governed by an adiabatic invariant which, in the quantum language, states that the number of photons is conserved. The conditions under which these laws hold are defined by conditions on three lengths: (1) the typical reduced wavelength of the waves,  $\lambda$ , as measured in a typical local Lorentz frame; (2) the typical length  $\mathcal{L}$  over which the amplitude, polarization, and wavelength of the waves vary, e.g., the radius of curvature of a wavefront; (3) the typical radius of curvature  $\mathcal{R}$  of the spacetime through which the waves propagate. Geometric optics is valid whenever the reduced wavelength is very short compared to each of the other scales. We should note that these laws, regardless of in the flat or curved spacetime, are derived from the usual free Maxwell field.

Recent investigations show that the interaction between the electromagnetic and gravitational fields could be appeared naturally in quantum electrodynamics with the photon effective action originating from one-loop vacuum polarization in curved spacetime [1]. This coupling model of the electromagnetic and gravitational fields is of great interest, since the appearance of cross-terms in the Lagrangian leads to the modifications of the coefficients involving the higher-order derivatives both in the Maxwell and Einstein equations. So the electromagnetic theories containing this coupling term have been studied extensively [1–4]. Drummond and Hathrell [1] argued that the quantum corrections introduce tidal gravitational forces on the photons which alter the characteristics of propagation, so that in some cases photons travel at speeds greater than unity. The one-loop effective action for QED in curved spacetime contains the equivalence principle violating interactions between the electromagnetic field and the spacetime curvature, and these interactions lead to a dependence of the photon velocity on the motion and polarization directions[1, 5–9]. By taking the analogy between the eikonal equation in geometric optics and the particle equation of motion, Ahmadi and Nouri-Zonoz [10] investigated the phase structure and the trajectory of the propagating photon semiclassically in the modified theory.

Although the coupling models for the electromagnetic and gravitational fields have been extensive used recently, they will result in “faster than light” photons and will invalid the Lorentz invariance and some physical laws. For this coupling models, the characteristics of propagation of the light are altered and the laws of geometric optics, in general, are invalid in the usual spacetime. The reason

is that the electromagnetic fields are described by the modified theories rather than the usual free Maxwell theory now. In this paper, we want to show that, by introducing an effective spacetime, the laws of geometric optics and focusing theorem of light rays are still valid for the modified theory.

The plan of the paper is as follows. In the next section we introduce a model for the interaction between the electromagnetic and gravitational fields, and obtain the motion equations of light by using the geometric optics approximation in the usual spacetime. In section III, by using the effective spacetime introduced in the appendix, we investigate the laws of geometric optics and the focusing theorem of light rays. We present our conclusions in the last section.

## II. MODIFIED THEORY FOR COUPLINGS BETWEEN ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

To study the interactions between the electromagnetic and gravitational fields, it is natural to consider the couplings between the Maxwell field and the Weyl tensor. Therefore, the electromagnetic theory with the Weyl correction has been studied extensively [11–16]. The action for a toy model of the electromagnetic field coupled to the Weyl tensor can be expressed as [16]

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{4} \left( F_{\mu\nu} F^{\mu\nu} - 4\alpha C^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \right], \quad (2.1)$$

where  $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$  is the usual electromagnetic tensor with a vector potential  $A_\mu$ ,  $\alpha$  is the coupling constant which has the dimension of the length-squared, and  $C_{\mu\nu\rho\sigma}$  is the Weyl tensor defined by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - (g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu}, \quad (2.2)$$

where  $g_{\mu\nu}$  represents the usual spacetime and the brackets around indices refer to the antisymmetric part. Varying the action (2.1) with respect to  $A_\mu$ , we obtain

$$\nabla_\mu (F^{\mu\nu} - 4\alpha C^{\mu\nu\rho\sigma} F_{\rho\sigma}) = 0. \quad (2.3)$$

Under the geometric optics assumption, we can set

$$A^\mu = (a^\mu + \varepsilon b^\mu + \varepsilon^2 c^\mu + \dots) e^{i\theta}, \quad (2.4)$$

where  $\theta$  is a phase, and  $\varepsilon = \lambda/L$  is a small dimensionless number with the minimum  $L$  of  $\mathfrak{L}$  and  $\mathfrak{R}$ . Substituting Eq. (2.4) into Eq. (2.3) and noting that the polarization vector is perpendicular to the rays, we get from the leading term (order  $1/\varepsilon^2$ )

$$k_\mu k^\mu a^\nu + 8\alpha C^{\mu\nu\rho\sigma} k_\mu k_\sigma a_\rho = 0. \quad (2.5)$$

If we take

$$a^\mu = af^\mu, \quad (2.6)$$

where  $f^\mu$  is a unit vector along  $a^\mu$ , Eq. (2.5) can be rewritten as

$$k_\mu(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu) = 0, \quad (2.7)$$

which tells us that the wave vector is not null now. That is to say, it is different from the null wave vector  $k_\mu k^\mu = 0$  in the usual free Maxwell theory.

From the subleading term (order  $1/\varepsilon$ ), we can obtain the propagation equation for the vector amplitude

$$2a_{;\mu}^\nu k^\mu + (k^\mu_{;\mu} a^\nu - a^\mu_{;\mu} k^\nu) = 8\alpha[(C^{\mu\nu\rho\sigma} a_\sigma k_\rho)_{;\mu} + C^{\mu\nu\rho\sigma} a_{\sigma;\rho} k_\mu], \quad (2.8)$$

which can be reformulated as a conservation law

$$[a^2(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu)]_{;\mu} = 0. \quad (2.9)$$

Consequently, the vector  $a^2(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu)$  can be considered as a conserved current, and the integral

$$\oint [a^2(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu)] d^3\Sigma_\mu, \quad (2.10)$$

has a fixed and unchanging value for each 3-volume cutting a given tube formed of light rays.

### III. GEOMETRIC OPTICS FOR MODIFIED THEORY IN EFFECTIVE SPACETIME

Eq. (2.7) shows that the wave vector is not null, which means that the first law of geometric optics is invalid for the modified theory in the usual spacetime. And Eq. (2.9) or (2.10) tells us that the conserved current  $a^2(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu)$  is directly related to the curvature of the usual spacetime in the modified theory. In the following we will prove that all laws of geometric optics are valid for the modified theory in the effective spacetime.

#### A. Light rays are null geodesics in the effective spacetime

In the usual free Maxwell theory, we do not need to distinguish between the photon 4-velocity  $U^\mu$ , i.e., the tangent vector to the light rays, and the wave vector  $k_\mu$  since they are simply related by

using the spacetime metric  $g^{\mu\nu}$  to raise the index, i.e.,  $U^\mu \sim k^\mu = g^{\mu\nu}k_\nu$  (the detail proof please see the footnote [17] ).

However, there is an important distinction in the modified theory. The wave vector  $k_\mu$ , defined as the derivative of the phase, is a covariant vector, whereas the 4-velocity  $U^\mu$  is a true contravariant vector. The relation between them is nontrivial.

In order to cast the results obtained in the modified theory into the familiar form, we introduce the new effective metric  $g_{\mu\nu}$  (please see appendix A) and define

$$\tilde{k}^\mu = g^{\mu\nu}k_\nu, \quad \tilde{k}_\mu = k_\mu. \quad (3.1)$$

Then, we can write the light cone condition (2.7) for the wave vector as the homogeneous form

$$\tilde{k}^\mu \tilde{k}_\mu = g^{\mu\nu} \tilde{k}_\mu \tilde{k}_\nu = g_{\mu\nu} \tilde{k}^\mu \tilde{k}^\nu = 0. \quad (3.2)$$

In the effective spacetime, we can prove that  $\tilde{k}^\mu \sim \tilde{U}^\mu$  with  $\tilde{U}^\mu = \frac{d\tilde{x}^\mu}{d\lambda}$ . Using the fact that  $\tilde{k}_\mu = \theta_{,\mu}$  is the gradient of a scalar and  $\theta_{;\mu\nu} = \theta_{;\nu\mu}$ , from  $\tilde{k}^\mu \tilde{k}_\mu = 0$  we can get the propagation equation for the wave vector

$$\tilde{k}_{\nu;\mu} \tilde{k}^\mu = 0. \quad (3.3)$$

On the other hand, the wave vector  $\tilde{k}_\nu$  satisfies the geodesic equation

$$\frac{D\tilde{k}_\nu}{d\lambda} = \frac{d\tilde{k}_\nu}{d\lambda} - \Gamma_{\nu\mu}^\sigma \tilde{k}_\sigma \tilde{U}^\mu = \tilde{k}_{\nu;\mu} \tilde{U}^\mu = 0. \quad (3.4)$$

By comparing Eqs. (3.3) and (3.4), we find that  $\tilde{k}^\mu \sim \tilde{U}^\mu$ .

From the geodesic equation (3.3) (or (3.4)) and the light cone condition (2.7) which are derived from the modified theory in the effective spacetime, we can state that the light rays are null geodesics now. That is to say, the first law of geometric optics is still valid for the modified theory in the effective spacetime.

## B. Law of conservation of photon number in the effective spacetime

By using Eq. (3.1), in the effective spacetime we can reformulate (2.9) as

$$(a^2 \tilde{k}^\mu)_{;\mu} = 0. \quad (3.5)$$

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[17] From  $k_\mu k^\mu = 0$  we can easily get  $k_{\nu;\mu} k^\mu = 0$ . On the other hand, the wave vector  $k_\nu$  satisfies the geodesic equation  $\frac{Dk_\nu}{d\lambda} = \frac{dk_\nu}{d\lambda} - \Gamma_{\nu\mu}^\sigma k_\sigma U^\mu = k_{\nu;\mu} U^\mu = 0$ . Therefore, we have  $k^\mu \sim U^\mu$ .

The vector  $a^2\tilde{k}^\mu$  is the conserved current now and the integral

$$\oint (a^2\tilde{k}^\mu) d^3\Sigma_\mu, \quad (3.6)$$

has a fixed and unchanging value for each 3-volume cutting a given tube formed of light rays. In the integral the tube must be so formed of rays that an integral of  $a^2\tilde{k}^\mu$  over the walls of the tube will give zero. What is the physical significance of Eq. (3.6)? To remain a purely classical one, it tells us that the “number of light rays” is conserved and  $a^2\tilde{k}^0$  is the “density of light rays” on an  $x^0 = \text{constant}$  hypersurface. However, for the concrete physical interpretation, we prefer to consider Eq. (3.6) as the law of conservation of photon number along this tube.

### C. Focusing theorem in the effective spacetime

For a bundle of rays lying in a surface of constant phase in the tube, we can take a tiny area of the two-dimensional cross section as  $\sigma$ . Then the conserved equation (3.6) can be reformulated as

$$a^2\sigma = \text{constant}, \quad (3.7)$$

i.e.,

$$\frac{d(a^2\sigma)}{d\lambda} = (a^2\sigma)_{;\mu}\tilde{k}^\mu = 0. \quad (3.8)$$

Using Eqs. (3.5) and (3.8), we can show that the area changes from point to point along the bundle of rays as a result of the rays which diverge away from each other or converge toward each other

$$\sigma_{;\mu}\tilde{k}^\mu = \sigma\tilde{k}_{;\mu}^\mu. \quad (3.9)$$

Then, we can find that

$$\frac{d^2\sqrt{\sigma}}{d\lambda^2} = -\left(\delta^2 + \frac{1}{2}\mathfrak{R}_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu\right)\sqrt{\sigma}, \quad (3.10)$$

with

$$\delta^2 = \frac{1}{2}\tilde{k}^{\mu;\nu}\tilde{k}_{\nu;\mu} - \frac{1}{4}(\tilde{k}_{;\mu}^\mu)^2, \quad (3.11)$$

where  $\mathfrak{R}_{\mu\nu}$  is the Ricci curvature tensor of the effective spacetime, and the quantity  $\delta$  is the shear of the bundle of rays because it measures the extent to which neighboring rays are sliding past each

other. Therefore, the focusing equation (3.10) tells us that the shear focuses on a bundle of rays, and the spacetime curvature also focuses on it if  $\Re_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu > 0$ , but defocuses on it if  $\Re_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu < 0$ .

Assuming that the energy density is non-negative in the effective spacetime, from the focusing equation (3.10) and the Einstein field equations we obtain the focusing theorem

$$\frac{d^2 \sqrt{\sigma}}{d\lambda^2} \leq 0, \quad (3.12)$$

which takes the same form as the usual free Maxwell theory.

#### IV. CONCLUSIONS

In the coupling models for the electromagnetic and gravitational fields, which are extensively used recently, the wave vector is not null and the conserved current  $a^2(k^\mu + 8\alpha C^{\mu\nu\rho\sigma}k_\sigma f_\rho f_\nu)$  is directly related to the curvature of the usual spacetime. That is to say, the first and third laws of geometric optics are invalid in the modified theory due to the “faster than light” photons and invalid of the Lorentz invariance.

By introducing the effective spacetime, we first show that the wave vector becomes null and obeys the geodesic equation, which means that the first law of geometric optics is valid for the modified theory in the effective spacetime. Noting that the integral  $\oint(a^2\tilde{k}^\mu)d^3\Sigma_\mu$  has a fixed and unchanging value for each 3-volume cutting a given tube formed of light rays, we then find that the amplitude is governed by an adiabatic invariant which, in the quantum language, states that the number of photons is conserved.

From the focusing equation we know that the shear focuses on a bundle of rays, and the spacetime curvature also focuses on it if  $\Re_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu > 0$ , but defocuses on it if  $\Re_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu < 0$ . Furthermore, by using the focusing equation and the Einstein field equations, we find that, if the energy density is non-negative, the focusing theorem of light rays for the modified theory in the effective spacetime takes the same form as usual.

#### Appendix A: Effective metrics

For simplicity and clarity, we just consider a general four-dimensional static and spherically symmetric spacetime. The metric can be expressed as

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2\theta d\phi^2), \quad (A1)$$

where  $g_{00}$ ,  $g_{11}$  and  $g_{22}$  are functions of the polar coordinate  $r$  only. For metric (A1) the appropriate basis 1-forms are

$$e^0 = e_t^0 dt, \quad e^1 = e_r^1 dr, \quad e^2 = e_\theta^2 d\theta, \quad e^3 = e_\phi^3 d\phi, \quad (\text{A2})$$

where the vierbein fields  $e_\mu^a$  defined by

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (\text{A3})$$

here  $\eta_{ab}$  is the Minkowski metric with the signature  $(-, +, +, +)$ . The vierbeins for metric (A1) are

$$e_\mu^a = \text{diag}(\sqrt{-g_{00}}, \sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}). \quad (\text{A4})$$

Using the antisymmetric combination of vierbeins [1]

$$U_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b, \quad (\text{A5})$$

the Weyl tensor can be rewritten as

$$C_{\mu\nu\rho\sigma} = \mathcal{A} \left( 2U_{\mu\nu}^{01} U_{\rho\sigma}^{01} - U_{\mu\nu}^{02} U_{\rho\sigma}^{02} - U_{\mu\nu}^{03} U_{\rho\sigma}^{03} + U_{\mu\nu}^{12} U_{\rho\sigma}^{12} + U_{\mu\nu}^{13} U_{\rho\sigma}^{13} - 2U_{\mu\nu}^{23} U_{\rho\sigma}^{23} \right), \quad (\text{A6})$$

with

$$\begin{aligned} \mathcal{A} = & -\frac{1}{12(g_{00}g_{11})^2 g_{22}} \left\{ \left[ g_{00}g_{11}g_{00}'' - \frac{1}{2}(g_{00}g_{11})' g_{00}' \right] g_{22} - g_{00}^2 g_{11}g_{22}'' \right. \\ & \left. + \frac{1}{2}(g_{00}^2 g_{11}' - g_{00}g_{11}g_{00}') g_{22}' - 2(g_{00}g_{11})^2 \right\}. \end{aligned} \quad (\text{A7})$$

In order to cast the equation of motion for the coupled photon propagation into a simplified form, we introduce three linear combinations of momentum components [1]

$$l_\nu = k^\mu U_{\mu\nu}^{01}, \quad n_\nu = k^\mu U_{\mu\nu}^{02}, \quad m_\nu = k^\mu U_{\mu\nu}^{23}, \quad (\text{A8})$$

together with the dependent combinations

$$\begin{aligned} p_\nu &= k^\mu U_{\mu\nu}^{12} = \frac{1}{e_0^0 k^0} \left( e_1^1 k^1 n_\nu - e_2^2 \tilde{k}^2 l_\nu \right), \\ r_\nu &= k^\mu U_{\mu\nu}^{03} = \frac{1}{e_2^2 \tilde{k}^2} \left( e_0^0 k^0 m_\nu + e_3^3 k^3 l_\nu \right), \\ q_\nu &= k^\mu U_{\mu\nu}^{13} = \frac{e_1^1 k^1}{e_2^2 \tilde{k}^2} m_\nu + \frac{e_1^1 e_3^3 k^1 k^3}{e_0^0 e_2^2 k^0 \tilde{k}^2} n_\nu - \frac{e_3^3 k^3}{e_0^0 k^0} l_\nu. \end{aligned} \quad (\text{A9})$$



The vectors  $l_\nu$ ,  $n_\nu$ ,  $m_\nu$  are independent and orthogonal to the wave vector  $k_\nu$ . Contracting Eq. (2.5) with  $l_\nu$ ,  $n_\nu$ ,  $m_\nu$  respectively, using the relation (A9) and introducing three independent polarisation components  $(a \cdot l)$ ,  $(a \cdot n)$ , and  $(a \cdot m)$ , we find that the equation of motion of the photon coupling with the Weyl tensor can be simplified as

$$\begin{pmatrix} K_{11} & 0 & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & 0 & K_{33} \end{pmatrix} \begin{pmatrix} a \cdot l \\ a \cdot n \\ a \cdot m \end{pmatrix} = 0, \quad (\text{A10})$$

with the coefficients

$$\begin{aligned} K_{11} &= (1 + 16\alpha\mathcal{A})(g^{00}k_0k_0 + g^{11}k_1k_1) + (1 - 8\alpha\mathcal{A})(g^{22}k_2k_2 + g^{33}k_3k_3), \\ K_{22} &= (1 - 8\alpha\mathcal{A})(g^{00}k_0k_0 + g^{11}k_1k_1 + g^{22}k_2k_2 + g^{33}k_3k_3), \\ K_{21} &= 24\alpha\mathcal{A}\sqrt{g^{11}g^{22}}k_1k_2, \quad K_{23} = 8\alpha\mathcal{A}\sqrt{-g^{00}g^{33}}k_0k_3, \\ K_{33} &= (1 - 8\alpha\mathcal{A})(g^{00}k_0k_0 + g^{11}k_1k_1) + (1 + 16\alpha\mathcal{A})(g^{22}k_2k_2 + g^{33}k_3k_3). \end{aligned} \quad (\text{A11})$$

The condition of Eq. (A10) with the non-zero solution is  $K_{11}K_{22}K_{33} = 0$ . The first root  $K_{11} = 0$  leads to the modified light cone

$$(1 + 16\alpha\mathcal{A})(g^{00}k_0k_0 + g^{11}k_1k_1) + (1 - 8\alpha\mathcal{A})(g^{22}k_2k_2 + g^{33}k_3k_3) = 0, \quad (\text{A12})$$

which corresponds to the case where the polarisation vector  $a_\mu$  is proportional to  $l_\mu$ . The second root  $K_{22} = 0$  corresponds to an unphysical polarisation and should be neglected. The third root is  $K_{33} = 0$ , i.e.,

$$(1 - 8\alpha\mathcal{A})(g^{00}k_0k_0 + g^{11}k_1k_1) + (1 + 16\alpha\mathcal{A})(g^{22}k_2k_2 + g^{33}k_3k_3) = 0, \quad (\text{A13})$$

which means that the vector  $a_\mu = \lambda m_\mu$ .

The above discussions show that the light cone condition depends on not only the coupling between the photon and the Weyl tensor, but also on the polarizations. We know from results (A12) and (A13) that the light cone condition is not modified for the radially directed photons (i.e.,  $k_2 = k_3 = 0$ ) but is modified for the orbital photons (i.e.,  $k_1 = k_2 = 0$ ), and the velocities of the photons for the two polarizations are different, i.e., the phenomenon of gravitational birefringence [1].

In order to cast the light cone condition (2.7) ( or the result (A12)) obtained in the modified theory into the familiar form, we introduce a new effective contravariant metric  $g^{\mu\nu}$  in which the nonzero

components are

$$\begin{aligned}
g^{00} &= (1 + 16\alpha\mathcal{A})g^{00}, \\
g^{11} &= (1 + 16\alpha\mathcal{A})g^{11}, \\
g^{22} &= (1 - 8\alpha\mathcal{A})g^{22}, \\
g^{33} &= (1 - 8\alpha\mathcal{A})g^{33},
\end{aligned} \tag{A14}$$

and its covariant metric is given by

$$\begin{aligned}
g_{00} &= \frac{1}{1 + 16\alpha\mathcal{A}}g_{00}, \\
g_{11} &= \frac{1}{1 + 16\alpha\mathcal{A}}g_{11}, \\
g_{22} &= \frac{1}{1 - 8\alpha\mathcal{A}}g_{22}, \\
g_{33} &= \frac{1}{1 - 8\alpha\mathcal{A}}g_{33}.
\end{aligned} \tag{A15}$$

Then, defining

$$\tilde{k}^\mu = g^{\mu\nu}k_\nu, \quad \tilde{k}_\mu = k_\mu, \tag{A16}$$

we can write the light cone condition (2.7) for the result (A12) into the homogeneous form

$$\tilde{k}^\mu \tilde{k}_\mu = g^{\mu\nu} \tilde{k}_\mu \tilde{k}_\nu = g_{\mu\nu} \tilde{k}^\mu \tilde{k}^\nu = 0. \tag{A17}$$

Similarly, in order to cast the light cone condition (2.7) for the result (A13) into the homogeneous form, we should introduce the effective contravariant metric

$$\begin{aligned}
g^{00} &= (1 - 8\alpha\mathcal{A})g^{00}, \\
g^{11} &= (1 - 8\alpha\mathcal{A})g^{11}, \\
g^{22} &= (1 + 16\alpha\mathcal{A})g^{22}, \\
g^{33} &= (1 + 16\alpha\mathcal{A})g^{33},
\end{aligned} \tag{A18}$$

and its covariant metric

$$\begin{aligned}
g_{00} &= \frac{1}{1 - 8\alpha\mathcal{A}}g_{00}, \\
g_{11} &= \frac{1}{1 - 8\alpha\mathcal{A}}g_{11}, \\
g_{22} &= \frac{1}{1 + 16\alpha\mathcal{A}}g_{22}, \\
g_{33} &= \frac{1}{1 + 16\alpha\mathcal{A}}g_{33}.
\end{aligned} \tag{A19}$$

Similarly, we can also cast the light cone condition (2.7) for the result (A13) into the homogeneous form.

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